

BPS INVARIANTS OF SEMI-STABLE SHEAVES ON RATIONAL SURFACES

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ABSTRACT. BPS invariants are computed, capturing topological invariants of moduli spaces of semi-stable sheaves on rational surfaces. For a suitable stability condition, it is proposed that the generating function of BPS invariants of a Hirzebruch surface Σ_ℓ takes the form of a product formula. BPS invariants for other stability conditions and other rational surfaces are obtained using Harder-Narasimhan filtrations and the blow-up formula. Explicit expressions are given for rank ≤ 3 sheaves on Σ_ℓ and the projective plane \mathbb{P}^2 . The applied techniques can be applied iteratively to compute invariants for higher rank.

1. INTRODUCTION

Topological invariants of moduli spaces of semi-stable sheaves on complex surfaces are a rich subject with links to many topics in physics and mathematics. Closely related topics in physics are gauge theory, instantons, electric-magnetic duality [26] and also (multi-center) black holes [4, 19, 20]. Instantons saturate the bound on their minimal action, the so-called Bogomolnyi-Prasad-Sommerfeld (BPS) bound. The prime interest of this article are topological invariants of moduli spaces of instantons, in particular their Poincaré polynomials, which are commonly referred to as “BPS invariants”. These invariants correspond also to (refined) supersymmetric indices enumerating supersymmetric or BPS states.

Instantons on complex surfaces correspond algebraically to semi-stable vector bundles and coherent sheaves [11, 5]. Generating functions of BPS invariants of sheaves on surfaces are computed for rank 1 by Göttsche [7] and rank 2 by Yoshioka [28, 29]. These generating functions lead to intriguing connections with (mock) modular forms [26, 8, 9, 2], which are a manifestation of electric-magnetic duality of the gauge theory [26]. Refs. [21, 23] compute BPS invariants for rank 3 sheaves with Chern classes such that stability coincides with semi-stability.

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The present article computes the BPS invariants of *semi-stable* sheaves with rank 3 on Hirzebruch surfaces Σ_ℓ and on the projective plane \mathbb{P}^2 , and explains how to generalize the computations to higher rank. The developed techniques can be applied straightforwardly to compute BPS invariants of the other rational surfaces. Although the extension from stable to semi-stable might seem a minor one, it requires to deal with various subtle but fundamental aspects of the moduli spaces of semi-stable sheaves, which could be neglected in Ref. [21]. Having resolved how to deal with these aspects for $r = 3$, the computations can in principle be extended to any rank.

This introduction continues with summarizing the contents of the paper, after recalling the computations in Ref. [21] which were inspired by [28, 29, 8, 9]. A crucial fact for the computations is that the blow-up $\phi : \tilde{\mathbb{P}}^2 \rightarrow \mathbb{P}^2$ is equal to the Hirzebruch surface $\Sigma_\ell \rightarrow C$ with $\ell = 1$. The fibre f and base C of Σ_ℓ are both isomorphic to \mathbb{P}^1 . Since no stable bundles on \mathbb{P}^1 exist, the BPS invariants of Σ_ℓ with polarization J chosen sufficiently close to f (a so-called “suitable” polarization, see Definition 5.1) vanish for sheaves with first Chern class c_1 and rank r such that $c_1 \cdot f \not\equiv 0 \pmod{r}$.

Wall-crossing then allowed to compute the BPS invariants for other choices of J . The BPS invariants of \mathbb{P}^2 were obtained from those of $\tilde{\mathbb{P}}^2$ by application of the blow-up formula [30, 9, 18], which is a simple relation between the generating functions of the invariants for \mathbb{P}^2 and $\tilde{\mathbb{P}}^2$. However, its original form is only valid for $\gcd(c_1 \cdot \phi^*H, r) = 1$ and $J = \phi^*H$, with H the hyperplane class of \mathbb{P}^2 .

The present paper describes how to deal with the cases when c_1 and r do not satisfy the constraints for vanishing of the BPS invariant or the blow-up formula. The formal theory of invariants of moduli spaces (or stacks) of semi-stable sheaves is developed by Joyce [12]. We will in particular use the notion of Poincaré functions for moduli stacks, which are a generalization of Poincaré polynomials of manifolds. The Poincaré function of a moduli stack is (conjecturally) related to the BPS invariant by (3.5). The BPS invariant is most natural from physics and leads to generating functions with modular properties.

The two novel ingredients of this paper are:

- (1) Eq. (4.2) which provides for any rank $r \geq 1$ the generating function of Poincaré functions of the moduli stack of sheaves on $\tilde{\mathbb{P}}^2$ whose restriction to the fibre f is semi-stable.

- (2) *Extended* Harder-Narasimhan filtrations $0 \subset F_1 \subset F_2 \subset \cdots \subset F_\ell = F$, whose definition (Def. 5.3) differs from the usual definition (5.2) of HN filtrations by allowing quotients $E_i = F_i/F_{i-1}$ with equal (Gieseker) stability $p_J(E_i, n) \succeq p_J(E_{i+1}, n)$. These filtrations in combination with the associated invariants (5.8) are particularly useful to compute generating functions of BPS invariants starting from Conjecture 4.2 and their changes accross walls of marginal stability.

To obtain the BPS invariants for a suitable polarization, one subtracts from Eq. (4.2) generating functions corresponding to extended HN-filtrations given by (5.8), analogous to the seminal papers about vector bundles on curves [10, 1]. Naturally, these techniques are also applicable to compute invariants of semi-stable invariants for other mathematical objects like vector bundles on curves and quivers. Also a solution to this recursive procedure is given analogous to Ref. [32]. Then repeated application of the formula for filtrations (which is equivalent with the wall-crossing formulas [16, 13]) gives the BPS invariants for other choices of the polarization.

Finally, the blow-up formula provides the invariants on \mathbb{P}^2 . The earlier mentioned condition $\gcd(c_1 \cdot \phi^* H, r) = 1$ is a consequence of the fact that the blow-up formula is applicable for the Poincaré functions $\mathcal{I}^\mu(\Gamma, w; J)$ with respect to μ -stability instead of the more refined Gieseker stability. However with the invariant for filtrations (5.8), it is straightforward to transform the BPS invariants $\Omega(\Gamma, w; J)$ to $\mathcal{I}^\mu(\Gamma, w; J)$ for μ -stability. The rational factors in Eq. (5.8) appear naturally in the relation between the generating functions of these invariants.

The paper illustrates in detail the above steps for sheaves with rank 2 and 3, and shows their agreement with various consistency conditions, e.g. the blow-up formula, integrality and $w \leftrightarrow w^{-1}$ symmetry of the Poincaré polynomial.

The outline of the paper is as follows. Section 2 reviews some necessary properties of sheaves on surfaces including stability conditions. Section 3 defines the invariants and generating functions. Section 4 presents the formula (4.2) for the contribution to the BPS invariants from sheaves whose restriction to the fibre is semi-stable. Then we continue with the computation of the invariants of $\tilde{\mathbb{P}}^2$ for any choice of polarization in Section 5. To this end an invariant is associated to a filtration with prescribed topological classes of the quotients. Finally Section 6 computes the generating function for sheaves on \mathbb{P}^2 with $(r, c_1) = (3, 0)$.

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2. SHEAVES ON SURFACES

We consider sheaves on a smooth projective surface S . The Chern character of the sheaf F is given by $\text{ch}(F) = r(F) + c_1(F) + \frac{1}{2}c_1(F)^2 - c_2(F)$ in terms of the rank $r(F)$ and its Chern classes $c_1(F)$ and $c_2(F)$. The vector $\Gamma(F)$ parametrizes in the following the topological classes of the sheaf $\Gamma(F) := (r(F), \text{ch}_1(F), \text{ch}_2(F))$. Other frequently occurring quantities are the determinant $\Delta(F) = \frac{1}{r(F)}(c_2(F) - \frac{r(F)-1}{2r(F)}c_1(F)^2)$, and $\mu(F) = c_1(F)/r(F) \in H^2(S, \mathbb{Q})$.

Given a filtration $0 \subset F_1 \subset \dots \subset F_\ell = F$, let $E_i = F_i/F_{i-1}$ and $\Gamma_i = \Gamma(E_i)$. The discriminant of F is given in terms of the subobjects and quotients by:

$$(2.1) \quad \Delta(\Gamma(F)) = \sum_{i=1}^{\ell} \frac{r(E_i)}{r(F)} \Delta(E_i) - \frac{1}{2r(F)} \sum_{i=2}^{\ell} \frac{r(F_i)r(F_{i-1})}{r(E_i)} (\mu(F_i) - \mu(F_{i-1}))^2.$$

We are interested in the moduli space (or moduli stack) of semi-stable sheaves with respect to Gieseker stability, but also the coarser μ -stability appears in order to apply the blow-up formula. To define these two stability conditions, let $C(S) \subset H^2(S, \mathbb{R})$ be the ample cone of S , and the (reduced) Hilbert polynomial $p_J(F, n) = \chi(F \otimes J^n)/r(F)$. For a surface S , we have [5]:

$$(2.2) \quad p_J(F, n) = J^2 n^2 / 2 + \left(\frac{c_1(F) \cdot J}{r(F)} - \frac{K_S \cdot J}{2} \right) n + \frac{1}{r(F)} \left(\frac{c_1(F)^2 - K_S \cdot c_1(F)}{2} - c_2(F) \right) + \chi(\mathcal{O}_S).$$

Note that this function can be obtained from the physical central charge as in [4, 19]. In the large volume limit, the stability condition asymptotes to the lexicographic ordering of polynomials based on their coefficients. This ordering is denoted by \prec . Then,

Definition 2.1. *A torsion free sheaf F is Gieseker stable (respectively semi-stable) if for every subsheaf $F' \subsetneq F$, $p_J(F', n) \prec p_J(F, n)$ (respectively $p_J(F', n) \preceq p_J(F, n)$).*

and

Definition 2.2. *Given a choice $J \in C(S)$, a torsion free sheaf F is called μ -stable if for every subsheaf $F' \subset F$, $\mu(F') \cdot J < \mu(F) \cdot J$, and μ -semi-stable if for every subsheaf F' , $\mu(F') \cdot J \leq \mu(F) \cdot J$.*

Thus μ -stability is a coarser stability condition than Gieseker stability, although the walls for both stability conditions are the same. We will find in Sections 5 and 6 that invariants based on Gieseker stability exhibit better integrality and polynomial properties than the ones based on μ -stability. On the other hand, operations like restriction to a curve and blowing-up a point of S are most natural for μ -semi-stable sheaves. A wall of marginal stability $W(F', F) \subset H^2(S, \mathbb{R})$ is the codimension 1 subspace of $C(S)$, such that $(\mu(F') - \mu(F)) \cdot J = 0$, but $(\mu(F') - \mu(F)) \cdot J \neq 0$ away from $W(F', F)$.

The moduli space $\mathcal{M}_J(\Gamma)$ of Gieseker semi-stable sheaves on S (with respect to the ample class J) whose rank and Chern classes are determined by Γ has complex dimension:

$$(2.3) \quad \dim_{\mathbb{C}} \mathcal{M}_J(\Gamma) = 2r^2\Delta - r^2\chi(\mathcal{O}_S) + 1.$$

Twisting a sheaf E by a line bundle \mathcal{L} is an isomorphism of moduli spaces. The Chern classes of the twisted sheaf $E' = E \otimes \mathcal{L}$ are:

$$\begin{aligned} r(E') &= r(E), \quad c_1(E') = c_1(E) + r(E)c_1(\mathcal{L}), \\ c_2(E') &= c_2(E) + (r(E) - 1)c_1(\mathcal{L})c_1(E) + c_1(\mathcal{L})^2 \frac{r(E)(r(E) - 1)}{2}. \end{aligned}$$

The discriminant remains invariant: $\Delta(E') = \Delta(E)$. This shows that it suffices to compute the generating functions for $c_1(E) \in H^2(S, \mathbb{Z}/r\mathbb{Z})$.

Determination of the generating function for $r \geq 2$ is complicated in general. To make progress, we specialize in the following to the set of ruled surfaces. A ruled surface is a surface $\Sigma_{g,\ell}$ together with a surjective morphism $\pi : \Sigma_{g,\ell} \rightarrow C$ to a curve C with genus g , such that the generic fibre is isomorphic to \mathbb{P}^1 for every point of C . Let f be the fibre of π , then $H_2(\Sigma_{g,\ell}, \mathbb{Z}) = \mathbb{Z}C \oplus \mathbb{Z}f$, with intersection numbers $C^2 = -\ell$, $f^2 = 0$ and $C \cdot f = 1$. The canonical class is $K_{\Sigma_{g,\ell}} = -2C + (2g - 2 - \ell)f$. The holomorphic Euler characteristic $\chi(\mathcal{O}_{\Sigma_{g,\ell}})$ is $1 - g$. An ample divisor $J \in C(\Sigma_{g,\ell})$ is parametrized by $J_{m,n} = m(C + \ell f) + nf$ with $m, n > 0$.

This article will further specialize to the Hirzebruch surfaces $\Sigma_{0,\ell} = \Sigma_{\ell}$, and in particular to Σ_1 . Besides being a ruled surface, Σ_1 is also the blow-up $\phi : \tilde{\mathbb{P}}^2 \rightarrow \mathbb{P}^2$ of the projective plane \mathbb{P}^2 . The exceptional divisor of ϕ is C , and the pullback of hyperplane class of \mathbb{P}^2

ϕ^*H equals $C + f$. Due to the simplicity of \mathbb{P}^2 , it is of intrinsic interest to determine the generating functions of its BPS invariants.

3. BPS INVARIANTS AND GENERATING FUNCTIONS

This section defines the generating functions of the BPS invariants and discusses some of its properties. Although this article computes the BPS invariants and shows that they satisfy many desired and expected properties, their precise definition is still not completely settled. Physically the BPS invariant is given by a weighted sum over the BPS Hilbert space $\mathcal{H}(\Gamma, J)$. We choose to divide by a factor $w - w^{-1}$ which appears physically as an overall sum of oscillators. The BPS invariant arises by considering topologically twisted $\mathcal{N} = 4$ Yang-Mills on the surface S [26]. The path integral of this theory localizes on the BPS solutions, in particular instantons, due to the topologically twisted supersymmetry [26]. The path integral shows in particular that the (numerical) BPS invariant corresponds to the Euler number of the BPS moduli space.

Alternatively one can consider the $\mathcal{N} = 2$ supersymmetric theory in $\mathbb{R}^{3,1}$ obtained from the compactification of IIA theory on the non-compact Calabi-Yau $\mathcal{O}(K_S) \rightarrow S$. From this last perspective the BPS invariant can be refined with an additional parameter w [6]:

$$(3.1) \quad \Omega(\Gamma, w; J) = \frac{\text{Tr}_{\mathcal{H}(\Gamma, J)} 2^{\hat{J}_3} (-1)^{2\hat{J}_3} (-w)^{2\hat{I}_3 + 2\hat{J}_3}}{(w - w^{-1})^2},$$

with \hat{J}_3 a generator of the $SU(2) \cong \text{Spin}(3)$ group arising from rotations in $\mathbb{R}^{3,1}$, and \hat{I}_3 is a generator of the $SU(2)_R$ R -symmetry group. BPS representations have the form $[(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})] \otimes \omega$ with $\omega = (j, j')$ a vacuum representation of $\text{Spin}(3) \oplus SU(2)_R$ with spins j and j' . One factor of $w - w^{-1}$ in the denominator will vanish due to the factor $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ (the half-hypermultiplet) present for every BPS state [6]. Since $\Omega(\Gamma, w; J)$ is thus essentially an $SU(2)$ character, this shows that $\Omega(\Gamma, w; J)$ is a polynomial divided by $w - w^{-1}$; the polynomial has integer coefficients and is invariant under $w \leftrightarrow w^{-1}$. The positivity conjectures of Ref. [6] assert furthermore that the coefficients are positive. Many formulas take a simpler form in terms of the rational BPS invariants given by:

$$(3.2) \quad \bar{\Omega}(\Gamma, w; J) := \sum_{m|\Gamma} \frac{\Omega(\Gamma/m, -(-w)^m; J)}{m}.$$

They appear in fact rather natural from the physical perspective [22, 14].

The $\mathcal{N} = 2$ picture shows that the refined BPS invariant provides more information than the Euler number of the moduli space. The w -expansion is expected to give the Poincaré polynomial of the BPS moduli space. To make this more precise, we let $\mathcal{M}_J(\Gamma)$ be the suitably compactified moduli space of semi-stable sheaves on S with topological classes Γ and for polarization $J \in C(S)$, i.e. the Gieseker-Maruyama compactification. If semi-stable is equivalent to stable the moduli space is smooth and one can take as mathematical definition of the BPS invariant

$$(3.3) \quad \Omega(\Gamma, w; J) := \frac{w^{-\dim_{\mathbb{C}} \mathcal{M}_J(\Gamma)}}{w - w^{-1}} p(\mathcal{M}_J(\Gamma), w),$$

where $p(X, w) = \sum_{i=0}^{2\dim_{\mathbb{C}}(X)} b_i(X) w^i$ is the Poincaré polynomial of X , with $b_i(X)$ the Betti numbers $b_i(X) = \dim H^i(X, \mathbb{Z})$. Eq. (2.3) provides us with the degree of $p(\mathcal{M}_J(\Gamma), w)$, and since $\mathcal{M}_J(\Gamma)$ is compact, orientable and without boundary $b_i(X) = b_{2\dim_{\mathbb{C}}(X)-i}(X)$.

If semi-stable is not equivalent to stable, $\mathcal{M}_J(\Gamma)$ contains singularities due to non-trivial automorphisms of the sheaves, and the mathematical definition of the BPS invariant is not clear. The proper way to deal with the singularities is to consider instead of the moduli space $\mathcal{M}_J(\Gamma)$ the moduli stack $\mathfrak{M}_J(\Gamma)$. Let $\mathcal{I}(\Gamma, w; J)$ be the virtual Poincaré function of the stack $\mathfrak{M}_J(\Gamma, w)$ of semi-stable sheaves [12] (see also [16, 24]). The virtual Poincaré function $\mathcal{I}(\Gamma, w; J)$ is a rational function in w and a natural generalization of the Poincaré polynomial of smooth projective varieties to stacks.

The seminal papers [7, 28, 29] compute moduli space invariants by explicitly counting sheaves on the surface S defined over a finite field \mathbb{F}_s with s elements. The Poincaré function $\mathcal{I}(\Gamma, s^{\frac{1}{2}}; J)$ is upto an overall monomial computed by:

$$(3.4) \quad \sum_{E \in M_J(\Gamma, \mathbb{F}_s)} \frac{1}{\#\text{Aut}(E)},$$

where $M_J(\Gamma, \mathbb{F}_s)$ is the set of semi-stable sheaves with characteristic classes Γ . The Weil conjectures imply that the expansion coefficients in s are the Betti numbers of the moduli spaces. The parameters s is related to the w in this paper by $s = w^2$. Eq. (3.4) shows that poles of $\mathcal{I}(\Gamma, w; J)$ in w appear when the sheaves have non-trivial automorphism groups. If semi-stable is equivalent to stable $\mathcal{I}(\Gamma, w; J) = \Omega(\Gamma, w; J)$; the factor $w - w^{-1}$ is due to the automorphisms which are multiplication by \mathbb{C}^* . The automorphism group of semi-stable and unstable bundles or sheaves is in general $GL(n)$, whose number of elements over \mathbb{F}_s is $(1 - s)(1 - s^2) \dots (1 - s^n)$ and thus lead to higher order poles.

If semi-stable is not equivalent to stable, the following assumes (and gives evidence) that $\mathcal{I}(\Gamma, w; J)$ is related to the multi-cover invariants $\bar{\Omega}(\Gamma, w; J)$ by:

$$(3.5) \quad \mathcal{I}(\Gamma, w; J) = \sum_{\substack{\Gamma_1 + \dots + \Gamma_\ell = \Gamma \\ p_J(\Gamma_i, n) = p_J(\Gamma, n) \text{ for } i=1, \dots, \ell}} \frac{1}{\ell!} \prod_{i=1}^{\ell} \bar{\Omega}(\Gamma_i, w; J),$$

with inverse relation:

$$(3.6) \quad \bar{\Omega}(\Gamma, w; J) = \sum_{\substack{\Gamma_1 + \dots + \Gamma_\ell = \Gamma \\ p_J(\Gamma_i, n) = p_J(\Gamma, n) \text{ for } i=1, \dots, \ell}} \frac{(-1)^{\ell+1}}{\ell} \prod_{i=1}^{\ell} \mathcal{I}(\Gamma_i, w; J),$$

Note that $\mathcal{I}(\Gamma, w^{-1}; J) \neq -\mathcal{I}(\Gamma, w; J)$ and that $\mathcal{I}(\Gamma, w; J)$ might have higher order poles in w as discussed above. The invariants $\mathcal{I}(\Gamma, w; J)$ are not so natural from the perspective of modular generating functions of BPS invariants, since the generating functions of $\bar{\Omega}(\Gamma, w; J)$ have weight $-b_2(S)/2$. However, the product formula of Conjecture 4.1 is a generating function for $\mathcal{I}(\Gamma, w; f)$ and invariants $\mathcal{I}(\Gamma, w; J)$ with respect to μ -stability instead of Gieseker stability appear in the blow-up formula in Section 6.

Since the invariants $\mathcal{I}(\Gamma, w; J)$ are not so natural for generating functions, the following will mainly work with the invariants $\Omega(\Gamma, w; J)$ and $\bar{\Omega}(\Gamma, w; J)$. This requires to multiply by factorials due to Eq. (3.5). Interestingly, these naturally combine generating functions of Gieseker unstable and strictly semi-stable sheaves.

We will find that $\Omega(\Gamma, w; J)$ indeed has the form (3.3), with the expected integrality properties. This generalizes Refs. [15, 29] where it is shown that $\bar{\Omega}(\Gamma, w; J)$ computes the intersection cohomology of the singular moduli space $\mathcal{M}_J(\Gamma)$ for $r = 2$. Intersection cohomology is a cohomology theory for manifolds with singularities which satisfies Poincaré duality if the manifolds are complex and compact. It is therefore natural to expect that the BPS invariants for $r \geq 3$ also provide Betti numbers of intersection cohomology groups. This issue is left for further research.

We continue now by defining the generating function $h_{r, c_1}(z, \tau; S, J)$ of $\bar{\Omega}(\Gamma, w; J)$:

$$(3.7) \quad h_{r, c_1}(z, \tau; S, J) = \sum_{c_2} \bar{\Omega}(\Gamma, w; J) q^{r\Delta(\Gamma) - \frac{r\chi(S)}{24}}.$$

where $q := e^{2\pi i\tau}$, with $\tau \in \mathcal{H}$ and $w := e^{2\pi iz}$ with $z \in \mathbb{C}$. Since twisting by a line bundle (2.4) is an isomorphism of moduli spaces, it suffices to compute $h_{r, c_1}(z, \tau; S, J)$ for $c_1 \in H_2(S, \mathbb{Z}/r\mathbb{Z})$. The expansion parameter t for c_2 in Refs. [7, 28, 29] is related to q by $q = s^r t$.

The generating function $h_{1,c_1}(z, \tau; S)$ depends only on $b_2(S)$ for S a smooth projective surface with $b_1(S) = b_3(S) = 0$ [7]:

$$(3.8) \quad h_{1,c_1}(z, \tau; S) = \frac{i}{\theta_1(2z, \tau) \eta(\tau)^{b_2(S)-1}},$$

where the Dedekind eta function $\eta(\tau)$ and Jacobi theta function $\theta_1(z, \tau)$ are defined by:

$$\begin{aligned} \eta(\tau) &:= q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \\ \theta_1(z, \tau) &:= iq^{\frac{1}{8}} (w^{\frac{1}{2}} - w^{-\frac{1}{2}}) \prod_{n \geq 1} (1 - q^n)(1 - wq^n)(1 - w^{-1}q^n). \end{aligned}$$

The dependence on J is omitted in Eq. (3.8), since all rank 1 torsion free sheaves are stable throughout $C(S)$. Similarly, J is omitted in the following from $h_{r,c_1}(z, \tau; \mathbb{P}^2, J)$, since $b_2(\mathbb{P}^2) = 1$ and therefore the BPS invariants do not vary as function of J . For clarity of exposition, Σ_ℓ is omitted from the arguments of $h_{r,c_1}(z, \tau; \Sigma_\ell, J)$.

4. RESTRICTION TO THE FIBRE OF HIRZEBRUCH SURFACES

This subsection deals with the set $M_f(\Gamma)$ of sheaves whose restriction to the (generic) fibre f of $\pi : \Sigma_\ell \rightarrow C$ is semi-stable. Since $f \cong \mathbb{P}^1$ all vector bundles on f are isomorphic to a sum of line bundles, and therefore the restriction of a sheaf to the fibre can only be semi-stable if its degree $d(E|_f) = c_1 \cdot f$ equals 0 mod r .¹ For these Chern classes the $\Omega(\Gamma; w, J) \neq 0$ for $J = J_{0,1}$ or for a suitable polarization $J_{\varepsilon,1}$ (see Definition 5.1).

The BPS invariants for $J_{\varepsilon,1}$ can be computed from the Poincaré function $\mathcal{I}(\Gamma, w; f)$ of the moduli stack $\mathfrak{M}_f(\Gamma)$ of sheaves whose restriction to f is semi-stable. We define the generating function $H_r(z, \tau; f)$ of $\mathcal{I}(\Gamma, w; f)$ by:

$$(4.1) \quad H_r(z, \tau; f) := \sum_{c_2} \mathcal{I}(\Gamma, w; f) q^{r\Delta(\Gamma) - \frac{\chi(S)}{24}}.$$

The following conjecture gives $H_r(z, \tau; f)$ for any $r \geq 1$:

Conjecture 4.1. *The function $H_r(z, \tau; f)$ is given by:*

$$(4.2) \quad H_r(z, \tau; f) = \begin{cases} \frac{i(-1)^{r-1} \eta(\tau)^{2r-3}}{\theta_1(2z, \tau)^2 \theta_1(4z, \tau)^2 \dots \theta_1((2r-2)z, \tau)^2 \theta_1(2rz, \tau)}, & \text{if } c_1 \cdot f = 0 \pmod{r}, \quad r \geq 1 \\ 0, & \text{if } c_1 \cdot f \neq 0 \pmod{r}, \quad r > 1. \end{cases}$$

¹Recall that a vector bundle F of rank r and degree d on a curve C is stable (respectively semi-stable) if for every subbundle $F' \subsetneq F$ (with rank r' and degree d') $d'/r' < d/r$ (respectively $d'/r' \leq d/r$).

Conjecture 4.1 is obviously in agreement with Eq. (3.8) for $r = 1$. Ref. [29] proved the conjecture for $(r, c_1) = (2, f)$, which is now briefly recalled. Ref. [29] considers the ruled surface $\tilde{\mathbb{P}}^2$ over a finite field \mathbb{F}_s , and utilizes the fact that any vector bundle in F can be obtained from $\pi^*\pi_*F$, which is a vector bundle on $\tilde{\mathbb{P}}^2$ supported on C , by successive elementary transformations.

An elementary transformation is defined by [11]:

Definition 4.2. *Let D be an effective divisor on the surface X . If F and G are vector bundles on S and D respectively, then a vector bundle F' on S is obtained by an elementary transformation of F along G if there exists an exact sequence:*

$$(4.3) \quad 0 \rightarrow F' \rightarrow F \rightarrow i_*G \rightarrow 0,$$

where i denotes the embedding $D \subset S$.

This shows that the contribution to $h_{2,c_1}(z, \tau; J)$ from $M_f(\Gamma)$ is the product of the the total set of vector bundles on C , multiplied by the number of elementary transformations.

The total set of vector bundles with $r = 2$ on C is enumerated by:

$$(4.4) \quad \frac{s^{-3}}{1-s} \zeta_C(2)$$

with $\zeta_C(n)$ the zeta function of the Riemann surface C :

$$(4.5) \quad \zeta_C(n) = \frac{\prod_{j=1}^{2g} (1 - \omega_j s^{-n})}{(1 - s^{-n})(1 - s^{1-n})}.$$

Multiplication by the factor due to elementary transformations gives [29]:

$$(4.6) \quad \sum_{c_2} \sum_{E \in M_f(2, mf, c_2)} \frac{t^{c_2}}{\#\text{Aut}(E)} = \frac{s^{-3}}{1-s} \zeta_C(2) \prod_{a \geq 1} Z_s(S, s^{2a-2}t^a) Z_s(S, s^{2a}t^a),$$

with $Z_s(S, t)$ the zeta function of the surface S :

$$(4.7) \quad Z_s(S, t) = \frac{1}{(1-t)(1-st)^{b_2(S)}(1-s^2t)}.$$

The parameter substitutions $q = s^r t$ and $w^2 = s$ give then Eq. (4.2) (upto an overall monomial in w and q). It is conceivable that Conjecture 4.1 for $r > 2$ can be proven in a similar manner. The following sections show that at least for $r = 3, 4$, it is consistent with various other results. Moreover, it continues to hold for the other Hirzebruch surfaces with $\ell \geq 0$.

As an aside, we mention that the Poincaré function of the stack of vector bundles on a curve C_g with genus g is given in terms of w by [10, 1]:

$$(4.8) \quad H_r(z; C_g) := -w^{r^2(1-g)} \frac{(1 + w^{2r-1})^{2g}}{1 - w^{2r}} \prod_{j=1}^{r-1} \frac{(1 + w^{2j-1})^{2g}}{(1 - w^{2j})^2}.$$

The first term in the q -expansion of Eq. (4.2) starts with Eq. (4.8) for $g = 0$. In fact, one could view $H_r(z, \tau; f)$ as an extension of $H_r(z; C_g)$ to a modular infinite product. It is conceivable that the generalization of Eq. (4.2) to ruled surfaces with $g > 0$ is similarly related to $H_r(z; C_g)$. For example, this gives for $r = 1$:

$$(4.9) \quad h_{1,c_1}(z, \tau; \Sigma_{g,\ell}) = \frac{i \theta_1(z + \frac{1}{2}, \tau)^{2g}}{\theta_1(2z, \tau) \eta(\tau)^{1+2g}}.$$

5. BPS INVARIANTS OF HIRZEBRUCH SURFACES

5.1. BPS invariants for a suitable polarization. This subsection computes for $c_1 \cdot f = 0 \bmod r$ the BPS invariants of Σ_ℓ for a polarization $J \in C(\Sigma_\ell)$ sufficiently close to $J_{0,1} = f$. The BPS invariants are for this choice of J independent of ℓ . “Sufficiently close” depends on the topological classes of the sheaf; it is known as a Γ -suitable polarization:

Definition 5.1. *A polarization J is called Γ -suitable if and only if:*

- J does not lie on a wall for $\Gamma = (r, \text{ch}_1, \text{ch}_2)$ and,
- for any semi-stable subsheaf $F' \subset F$ with $\Gamma(F) = \Gamma$, $(\mu(F') - \mu(F)) \cdot f = 0$ or $(\mu(F') - \mu(F)) \cdot f$ and $(\mu(F') - \mu(F)) \cdot J$ have the same sign.

We will keep the dependence on the Chern classes implicit in the following and denote a suitable polarization by $J_{\varepsilon,1}$ with ε positive but sufficiently small. From the definition follows that if $J_{\varepsilon,1}$ is an $\Gamma(F)$ -suitable polarization, and $F|_f$ is unstable, then F is μ -unstable. Thus we need to subtract from $M_f(\Gamma)$, i.e. the set of sheaves with topological classes Γ whose restriction to the fibre f is semi-stable, the set of sheaves which are semi-stable for $J = f$ but Gieseker unstable for $J_{\varepsilon,1}$. We continue by explaining this for $r = 2$. Then the general formula is proposed for the invariant enumerating extended HN-filtrations in a suitable way, which is then illustrated for $r = 3$.

A crucial tool to obtain the invariants enumerating semi-stable sheaves are Harder-Narasimhan filtrations [10], which can be defined for either Gieseker or μ -stability. To define these filtrations, let φ denote either Gieseker, $\varphi(F) = p_J(F, n)$, or μ -stability, $\varphi(F) = \mu(F) \cdot J$. Then:

Definition 5.2. A Harder-Narasimhan filtration (HN-filtration) with respect to the stability condition φ is a filtration $0 \subset F_1 \subset F_2 \subset \cdots \subset F_\ell = F$ of the sheaf F such that the quotients $E_i = F_i/F_{i-1}$ are semi-stable with respect to φ and satisfy $\varphi(E_i) > \varphi(E_{i+1})$ for all i .

Since μ -stability is coarser than Gieseker stability, the length $\ell_G(F)$ of the HN-filtration with respect to Gieseker stability is in general larger than the length $\ell_\mu(F)$ of its HN-filtration with respect to μ -stability.

The Poincaré function of the stack of HN-filtrations with respect to Gieseker stability and prescribed $\Gamma_i = \Gamma(E_i)$ is [30]:

$$(5.1) \quad w^{-\sum_{i < j} r_i r_j (\mu_j - \mu_i) \cdot K_S} \prod_{i=1}^{\ell} \mathcal{I}(\Gamma, w; J),$$

where $r_i r_j (\mu_j - \mu_i) \cdot K_S$ is the Euler form for semi-stable sheaves on the projective surface S . One could define a similar function for the stack of filtrations with respect to μ -stability.

For $(r, c_1) = (2, f)$, the only HN-filtrations have length $\ell_G = 2$. Denoting $c_1(E_2) = bC - af$, and thus $c_1(E_1) = -bC + (a+1)f$, one easily verifies that the HN-filtrations correspond to $a \geq 0$ and $b = 0$. Since $b = 0$ the dependence of K_S in Eq. (5.1) does not lead to a dependence on ℓ . Using that Eq. (3.8) is also the generating function of $\mathcal{I}(\Gamma, w; J)$ for $r = 1$, Eq. (5.1) becomes:

$$(5.2) \quad \sum_{a \geq 0} w^{-2(2a+1)} h_{1,0}(z, \tau)^2 = -\frac{w^2}{1-w^4} h_{1,0}(z, \tau)^2,$$

where we assumed $|w| > 1$. Subtracting this from Eq. (4.2) for $r = 2$ gives:

$$(5.3) \quad h_{2,f}(z, \tau; J_{\varepsilon,1}) = \frac{-1}{\theta_1(2z, \tau)^2 \eta(\tau)^2} \left(\frac{i \eta(\tau)^3}{\theta_1(4z, \tau)} + \frac{w^2}{1-w^4} \right),$$

which is easily verified to enumerate invariants $\bar{\Omega}(\Gamma; J_{\varepsilon,1})$ satisfying the expected properties mentioned below Eq. (3.3).

For $(r, c_1) = (2, 0)$, the HN-filtrations with $\ell_G = 2$ split naturally in two subsets: the first set has length $\ell_\mu = 2$ with respect to μ -stability, and the second set has $\ell_\mu = 1$. Similarly to (5.2), the first set gives rise to:

$$(5.4) \quad -\frac{1}{1-w^4} h_{1,0}(z, \tau)^2,$$

and the second set to:

$$(5.5) \quad \frac{1}{2} h_{1,0}(z, \tau)^2 - \frac{1}{2} \sum_{n \geq 0} \Omega((1, 0, n); w)^2 q^{2n},$$

where the second term subtracts from the first the Gieseker semi-stable sheaves which should not be subtracted from $H_2(z, \tau; f)$. Subtraction of Eqs. (5.4) and (5.5) from $H_2(z, \tau; f)$ gives the generating function of $\mathcal{I}((2, 0, c_2), w; J)$, which corresponds by Eq. (3.5) to:

$$(5.6) \quad h_{2,0}(z, \tau; J_{\varepsilon,1}) = \frac{-1}{\theta_1(2z, \tau)^2 \eta(\tau)^2} \left(\frac{i \eta(\tau)^3}{\theta_1(4z, \tau)} + \frac{1}{1 - w^4} - \frac{1}{2} \right),$$

Again one can verify that the invariants satisfy the expected integrality properties. Ref. [29] computes the Betti numbers of the intersection cohomology of the singular moduli spaces and arrives at the same generating function (5.6).

The Betti numbers for the intersection cohomology of the moduli space of semi-stable vector bundles on Riemann surfaces were earlier computed in Ref. [15]. The above procedure gives these Betti numbers with much less effort. For example, one can easily verify that

$$(5.7) \quad H_2(z, C_g) + \left(\frac{1}{1 - w^4} - \frac{1}{2} \right) H_1(z, C_g)^2,$$

with $H_r(z, C_g)$ as in Eq. (4.8), is equivalent with Proposition 5.9 in the Corrigendum to [15].

Since the invariants $\mathcal{I}(\Gamma, w; J)$ are not so compatible with modular generating functions for $r \geq 2$, it is useful to work as much as possible with the invariants $\bar{\Omega}(\Gamma, w; J)$. To this end an extension of the HN-filtration is necessary:

Definition 5.3. *An extended Harder-Narasimhan filtration (with respect to Gieseker stability) is a filtration $0 \subset F_1 \subset F_2 \subset \dots \subset F_\ell = F$ whose quotients $E_i = F_i/F_{i-1}$ are semi-stable and satisfy $p_J(E_i, n) \succeq p_J(E_{i+1}, n)$.*

From Eq. (3.5) follows that the natural invariant $\bar{\Omega}(\{\Gamma_i\}, w; J)$ associated to the stack $\mathfrak{M}(\{\Gamma_i\}, w)$ of extended HN-filtrations with prescribed Chern classes $\Gamma_i = \Gamma(E_i)$ is:

$$(5.8) \quad \bar{\Omega}(\{\Gamma_i\}; w, J) := \frac{1}{|\text{Aut}(\{\Gamma_i\}; J)|} w^{-\sum_{i < j} r_i r_j (\mu_j - \mu_i) \cdot K_S} \prod_{i=1}^{\ell} \bar{\Omega}(\Gamma_i, w; J).$$

The number $|\text{Aut}(\{\Gamma_i\}; J)|$ is equal to $\prod_a m_a!$, where m_a is the total number of quotients E_i with equal reduced Hilbert polynomial $p_J(E_a, n)$. Thus only for HN-filtrations $|\text{Aut}(\{\Gamma_i\}; J)| = 1$.

If the sum over all extended HN-filtrations contains a group E_i with equal $p_J(E_i, n)$ but unequal Γ_i , the factor $\frac{1}{|\text{Aut}(\{\Gamma_i\}; J)|}$ divides out a number of permutations. To avoid this overcounting, one could introduce a further ordering on the vectors Γ_i , which should be obeyed by the set of filtrations to be summed over. Then one would divide by $|\text{Aut}(\{\Gamma_i\})| = \prod_p n_p!$, where n_p is the number of equal vectors Γ_p appearing among the Γ_i , $i = 1, \dots, \ell$.

This is the origin of the “Boltzmann statistics” in wall-crossing formulas [22] in the work of Joyce [12].

The functions $h_{r,c_1}(z, \tau; J_{\varepsilon,1})$ with $c_1 \cdot f = 0 \pmod r$ are given by the recursive formula

$$(5.9) \quad h_{r,c_1}(z, \tau; J_{\varepsilon,1}) = H_r(z, \tau; f) - \sum_{\text{ch}_2} \sum_{\substack{\Gamma_1 + \dots + \Gamma_\ell = (r, c_1, \text{ch}_2) \\ p_J(\Gamma_i, n) \geq p_J(\Gamma_{i+1}, n), \ell > 1}} \bar{\Omega}(\{\Gamma_i\}; w, J_{\varepsilon,1}) q^{r\Delta(\Gamma) - \frac{r\chi(S)}{24}},$$

with $\Delta(\Gamma)$ given in terms of Γ_i by Eq. (2.1).

We continue by applying Eq. (5.8) to compute $h_{3,c_1}(z, \tau; J_{1,\varepsilon})$, with $c_1 = f$ and 0. One obtains:

Proposition 5.4.

$$(5.10) \quad h_{3,f}(z, \tau; J_{\varepsilon,1}) = \frac{i\eta(\tau)^3}{\theta_1(2z, \tau)^2 \theta_1(4z, \tau)^2 \theta_1(6z, \tau)} + \frac{w^2 + w^4}{1 - w^6} \frac{1}{\theta_1(2z, \tau)^3 \theta_1(4z, \tau)} - \frac{w^4}{(1 - w^4)^2} \frac{i}{\theta_1(2z, \tau)^3 \eta(\tau)^3},$$

$$(5.11) \quad h_{3,0}(z, \tau; J_{\varepsilon,1}) = \frac{i\eta(\tau)^3}{\theta_1(2z, \tau)^2 \theta_1(4z, \tau)^2 \theta_1(6z, \tau)} + \frac{1 + w^6}{1 - w^6} \frac{1}{\theta_1(2z, \tau)^3 \theta_1(4z, \tau)} - \left(\frac{w^4}{(1 - w^4)^2} + \frac{1}{3} \right) \frac{i}{\theta_1(2z, \tau)^3 \eta(\tau)^3}.$$

Proof. We start by proving Eq. (5.10). Denote the length of an extended HN-filtration by ℓ , its length with respect to μ -stability by ℓ_μ and Gieseker stability ℓ_G . We first consider the unstable filtrations with $\ell = \ell_\mu = 2$, and parametrize $c_1(E_2)$ by $bC - af$. These are parametrized by $a \geq 0$ and $b = 0$. There are four possibilities to be distinguished: whether $r(E_1) = 1$ or 2, and whether the quotient with rank 2 has $c_1 = 0$ or $f \pmod 2$. Adding up these contributions, one obtains:

$$(5.12) \quad -\frac{w^4 + w^8}{1 - w^{12}} h_{1,0}(z, \tau) h_{2,0}(z, \tau; J_{\varepsilon,1}) - \frac{w^2 + w^{10}}{1 - w^{12}} h_{1,0}(z, \tau) h_{2,f}(z, \tau; J_{\varepsilon,1}),$$

The filtrations with $\ell = 3$ consist of 3 subsets: one set with $\ell_\mu = 3$, one with $\ell_\mu = 2$ but $\ell_G = 3$, and one with $\ell_G = 2$. Parametrizing $c_1(E_i) = b_i C - a_i f$, the first set is parametrized by $a_i - a_{i+1} > 0$, $\sum_{i=1}^3 a_i = 1$ and $b_i = 0$. These are counted by:

$$(5.13) \quad \sum_{\substack{k_1, k_2 > 0 \\ k_2 = k_1 - 1 \pmod 3}} w^{-4(k_1 + k_2)} h_{1,0}(z, \tau)^3 = \frac{w^4}{(1 - w^4)(1 - w^{12})} h_{1,0}(z, \tau)^3.$$

For the second and third sets, one needs to distinguish between equality of the stability condition of E_2 with E_1 or E_3 . These two sets are enumerated by:

$$(5.14) \quad -\frac{1}{2} \frac{w^4 + w^8}{1 - w^{12}} h_{1,0}(z, \tau)^3.$$

Note that the factor $\frac{1}{|\text{Aut}(\{\Gamma_i\}, J_{1,\varepsilon})|}$ naturally combines the contributions of filtrations with $\ell_\mu < \ell$. Another observation is that the term $-\frac{1}{2}$ in the second factor of $h_{2,0}(z, \tau; J_{\varepsilon,1})$ (5.6) cancels against (5.14) in the total sum. After subtraction of the terms (5.12)-(5.14) from Eq. (4.2) for $r = 3$, and writing the whole series in terms of modular functions, one obtains (5.10).

For $(r, c_1) = (3, 0)$, one needs to subtract the following terms:

- due to unstable filtrations with $\ell = \ell_\mu = 2$:

$$-\frac{2}{1 - w^{12}} h_{1,0}(z, \tau) h_{2,0}(z, \tau; J_{\varepsilon,1}) - \frac{2w^6}{1 - w^{12}} h_{1,0}(z, \tau) h_{2,f}(z, \tau; J_{\varepsilon,1}),$$

- due to unstable filtrations with $\ell = 2$, $\ell_\mu = 1$ and $\ell_G = 1$ or 2 :

$$\frac{2}{2} h_{1,0}(z, \tau) h_{2,0}(z, \tau; J_{\varepsilon,1}),$$

- due to unstable filtrations with $\ell = \ell_\mu = 3$:

$$\frac{1 + w^{12}}{(1 - w^8)(1 - w^{12})} h_{1,0}(z, \tau)^3,$$

- due to unstable filtrations with $\ell = 3$, $\ell_\mu = 2$ and $\ell_G = 2$ or 3 :

$$-\frac{2}{2} \frac{1}{1 - w^{12}} h_{1,0}(z, \tau)^3,$$

- due to unstable filtrations with $\ell = 3$, $\ell_\mu = 1$ and $1 \leq \ell_G \leq 3$:

$$\frac{1}{6} h_{1,0}(z, \tau)^3.$$

Subtracting the terms above from (4.2) gives (5.11). Subtracting further $\frac{1}{3} h_{1,0}(3z, 3\tau) = \frac{i}{3\theta_1(6z, 3\tau)\eta(3\tau)}$ from (5.11) provides integer invariants in agreement with the definition (3.2). \square

The recursive procedure explained above can be solved, such that $h_{r,c_1}(z, \tau; J_{\varepsilon,1})$ can be directly expressed in terms of the $H_{r'}(z, \tau; f)$ with $r' \leq r$, without computing first the $h_{r',c_1}(z, \tau; J_{\varepsilon,1})$, and moreover giving more compact expressions. The solution follows from Ref. [32] (the solution to the recursion for vector bundles over Riemann surfaces) and Eq.

(3.6) one obtains:

$$\begin{aligned}
 h_{r,c_1}(z, \tau; J_{\varepsilon,1}) &= \sum_{\substack{(r_1, c_{1,1}) + \dots + (r_\ell, c_{1,\ell}) = (r, c_1), \\ \mu_i \cdot J_{\varepsilon,1} \geq \mu_{i+1} \cdot J_{\varepsilon,1}}} \frac{(-1)^{m-1}}{m} w^{-\sum_{i < j} r_i r_j (\mu_j - \mu_i) \cdot K_S} \prod_{i=1}^m H_{r_i}(z, \tau; f) \\
 (5.15) \quad &= \sum_{\substack{(r_1, a_1) + \dots + (r_\ell, a_\ell) = (r, c_1 \cdot C), \\ a_i \geq a_{i+1}}} \frac{(-1)^{m-1}}{m} w^{-2 \sum_{i < j} r_i r_j (a_j - a_i)} \prod_{i=1}^m H_{r_i}(z, \tau; f)
 \end{aligned}$$

This becomes after carrying out the sums over a_i [32]:

$$\begin{aligned}
 h_{r,-af}(z, \tau; J_{\varepsilon,1}) &= \sum_{\substack{(r_1, a_1) + \dots + (r_m, a_m) = (r, a) \\ a_i / r_i = a / r}} \frac{(-1)^{m-1}}{m} \\
 (5.16) \quad &\prod_{i=1}^m \left(\sum_{r_1 + \dots + r_\ell = r_i} \frac{w^{2M(r_1, \dots, r_\ell; a_i / r_i)}}{(1 - w^{2(r_1 + r_2)}) \dots (1 - w^{2(r_{\ell-1} + r_\ell)})} H_{r_1}(z, \tau; f) \dots H_{r_\ell}(z, \tau; f) \right),
 \end{aligned}$$

where

$$(5.17) \quad M(r_1, \dots, r_\ell; \lambda) = \sum_{j=1}^{\ell-1} (r_j + r_{j+1}) \{(r_1 + \dots + r_j) \lambda\},$$

with $\{\lambda\} := \lambda - \lfloor \lambda \rfloor$.

One can verify that Eq. (5.16) for $r = 3$ is in agreement with Eqs. (5.10) and (5.11). As an example we give here $h_{4,0}(z, \tau; J_{\varepsilon,1})$:

$$\begin{aligned}
 h_{4,0}(z, \tau) &= H_4(\tau, z; f) + \frac{1}{2} \frac{1 + w^8}{1 - w^8} H_2(\tau, z; f)^2 + \frac{1 + w^8}{1 - w^8} H_1(\tau, z; f) H_3(\tau, z; f) \\
 (5.18) \quad &+ \frac{1 - w^{16}}{(1 - w^4)(1 - w^6)^2} H_1(\tau, z; f)^2 H_2(\tau, z; f) + \frac{1}{4} \frac{1 - w^{16}}{(1 - w^4)^4} H_1(\tau, z; f)^4,
 \end{aligned}$$

which is to be compared with:

$$\begin{aligned}
h_{4,0}(z, \tau) = & H_4(\tau, z; f) - \left(-\frac{w^{12}}{(1-w^8)(1-w^{12})^2} + \frac{1}{2} \frac{1+w^{24}}{(1-w^{12})(1-w^{24})} \right. \\
& \left. - \frac{1}{3} \frac{1}{1-w^{24}} - \frac{1}{4} \frac{1}{1-w^{16}} + \frac{1}{24} \right) h_{1,0}(z, \tau)^4 \\
& - \left(\frac{2(1+w^{20})}{(1-w^{16})(1-w^{24})} + \frac{1+w^{24}}{(1-w^{12})(1-w^{24})} - \frac{2}{1-w^{24}} \right. \\
& \left. - \frac{1}{1-w^{16}} + \frac{1}{2} \right) h_{1,0}(z, \tau)^2 h_{2,0}(z, \tau; J_{\varepsilon,1}) \\
& - \left(\frac{2(w^{10}+w^{30})}{(1-w^{16})(1-w^{24})} + \frac{2w^{18}}{(1-w^{12})(1-w^{24})} \right) h_{1,0}(z, \tau)^2 h_{2,f}(z, \tau; J_{\varepsilon,1}) \\
& - \left(-\frac{1}{1-w^{16}} + \frac{1}{2} \right) h_{2,0}(z, \tau; J_{\varepsilon,1})^2 - \left(-\frac{w^8}{1-w^{16}} \right) h_{2,f}(z, \tau; J_{\varepsilon,1})^2 \\
& - \left(-\frac{2}{1-w^{24}} + 1 \right) h_{1,0}(z, \tau) h_{3,0}(z, \tau; J_{\varepsilon,1}) \\
& - \left(-\frac{2(w^8+w^{16})}{1-w^{24}} \right) h_{1,0}(z, \tau) h_{3,0}(z, \tau; J_{\varepsilon,1}).
\end{aligned}$$

5.2. Wall-crossing. This subsection explains how to compute $h_{r,c_1}(z, \tau; J)$ for generic choice of polarization J from the generating functions for $J = J_{\varepsilon,1}$. The BPS invariants $\Omega(\Gamma, w; J)$ for J differ in general from those for $J = J_{\varepsilon,1}$, since sheaves might become semi-stable or unstable by changing the polarization. The change of the BPS invariants depends on the choice of Hirzebruch surface Σ_ℓ . Knowing how $h_{r,c_1}(z, \tau; J)$ varies in $C(\Sigma_1)$ is particularly important for the computation of $h_{r,c_1}(z, \tau; \mathbb{P}^2)$ since the blow-up formula is to be applied for the polarization $J_{1,0} = \phi^* H$, where H is the hyperplane class of \mathbb{P}^2 (see the next section). The change of the invariants can be obtained recursively from Eq. (5.8) after determining which filtrations change from semi-stable to unstable or vice versa.

More quantitatively one has for J and J' sufficiently close:

$$\begin{aligned}
\Delta \bar{\Omega}(\Gamma, w; J \rightarrow J') = & \sum_{\substack{\Gamma = \Gamma_1 + \dots + \Gamma_\ell, \\ p_{J'}(\Gamma_i) \leq p_{J'}(\Gamma_{i+1}), \\ p_J(\Gamma_i) \geq p_J(\Gamma_{i+1})}} \frac{1}{|\text{Aut}(\{\Gamma_i\}; J)|} w^{-\sum_{i < j} r_i r_j (\mu_j - \mu_i) \cdot K_S} \prod_{i=1}^{\ell} \bar{\Omega}(\Gamma_i; w, J) \\
(5.19) \quad & - \sum_{\substack{\Gamma = \Gamma_1 + \dots + \Gamma_\ell, \\ p_{J'}(\Gamma_i) \geq p_{J'}(\Gamma_{i+1}), \\ p_J(\Gamma_i) \leq p_J(\Gamma_{i+1})}} \frac{1}{|\text{Aut}(\{\Gamma_i\}; J')|} w^{-\sum_{i < j} r_i r_j (\mu_j - \mu_i) \cdot K_S} \prod_{i=1}^{\ell} \bar{\Omega}(\Gamma_i; w, J'),
\end{aligned}$$

with $|\text{Aut}(\{\Gamma_i\}; J)|$ defined below Eq. (5.8). Note that the invariants are evaluated in the chamber where the filtration is unstable. This makes this formula an recursive formula as it requires knowledge of $\Omega(\Gamma_i, w; J')$, but since we are only interested in small rank this is not a serious obstacle. The recursion can be solved in various ways, e.g. [12, 13, 16] or the Higgs branch analysis of Ref. [22] based on Ref. [25].

Since generating functions capturing wall-crossing are already described in the literature, the explicit expressions of $h_{r,c_1}(z, \tau; J_{m,n})$ for $r = 2$ and 3, are presented here without further details. We have for $r = 2$ [28, 8]:

$$\begin{aligned} h_{2,\beta C-\alpha f}(z, \tau; J_{m,n}) &= h_{2,\beta C-\alpha f}(z, \tau; J_{\varepsilon,1}) + \\ &\quad \frac{1}{2} \sum_{a,b \in \mathbb{Z}} \frac{1}{2} \left(\text{sgn}((2b - \beta)n - (2a - \alpha)m) - \text{sgn}((2b - \beta) - (2a - \alpha)\varepsilon) \right) \\ &\quad \times \left(w^{-(\ell-2)(2b-\beta)-2(2a-\alpha)} - w^{(\ell-2)(2b-\beta)+2(2a-\alpha)} \right) q^{\frac{\ell}{4}(2b-\beta)^2 + \frac{1}{2}(2b-\beta)(2a-\alpha)} h_{1,0}(z, \tau)^2, \end{aligned}$$

and for $r = 3$ [21, 20]:

$$\begin{aligned} h_{3,\beta C-\alpha f}(z, \tau; J_{m,n}) &= h_{3,\beta C-\alpha f}(z, \tau; J_{\varepsilon,1}) + \\ &\quad \sum_{a,b \in \mathbb{Z}} \frac{1}{2} \left(\text{sgn}((3b - 2\beta)n - (3a - 2\alpha)m) - \text{sgn}((3b - 2\beta) - (3a - 2\alpha)\varepsilon) \right) \\ &\quad \times \left(w^{-(\ell-2)(3b-2\beta)-2(3a-2\alpha)} - w^{(\ell-2)(3b-2\beta)+2(3a-2\alpha)} \right) q^{\frac{\ell}{12}(3b-2\beta)^2 + \frac{1}{6}(3b-2\beta)(3a-2\alpha)} \\ &\quad \times h_{2,bC-\alpha f}(z, \tau; \Sigma_\ell, J_{|3b-2\beta|, |3a-2\alpha|}) h_{1,0}(z, \tau). \end{aligned}$$

6. BPS INVARIANTS OF \mathbb{P}^2

The Hirzebruch surface Σ_1 can be obtained as a blow-up $\phi : \Sigma_1 \rightarrow \mathbb{P}^2$ of the projective plane \mathbb{P}^2 . Interestingly, we can compute the BPS invariants of \mathbb{P}^2 from those of Σ_1 from the blow-up formula. This formula is a remarkable result which states that ratio of generating functions of BPS invariants of a surface S and its blow-up \tilde{S} is a (theta) function independent of S or J [30, 9, 18]. The underlying reason for this relation is that every semi-stable sheaf on \tilde{S} can be obtained from one on S by an elementary transformation along the exceptional divisor of the blow-up.

Two subtle issues of the blow-up formula are (Proposition 3.4 of [30]):

- the stability condition is μ -stability rather than Gieseker stability,
- it involves the virtual Poincaré functions $\mathcal{I}(\Gamma, w; J)$ of the moduli stack.

To take these two issues into account let $\bar{\Omega}^\mu(\Gamma, w; J)$ be the invariant enumerating μ -semi-stable sheaves which is obtained from $\bar{\Omega}^\mu(\Gamma, w; J)$ by addition of the Gieseker unstable sheaves

which are μ -semi-stable using Eq. (5.8). Moreover, let $\mathcal{I}^\mu(\Gamma, w; J)$ be the corresponding virtual Poincaré function with corresponding generating function $H_{r,c_1}^\mu(z, \tau; \tilde{S}, J)$.

The blow-up formula now reads [30]:

Proposition 6.1. *Let S be a smooth projective surface and $\phi : \tilde{S} \rightarrow S$ the blow-up at a non-singular point, with C_1 the exceptional divisor of ϕ . The generating functions $H_{r,c_1}^\mu(z, \tau; S, J)$ and $H_{r,c_1}^\mu(z, \tau; \tilde{S}, J)$ are related by the “blow-up formula”:*

$$H_{r,\phi^*c_1-kC_1}^\mu(z, \tau; \tilde{S}, \phi^*J) = B_{r,k}(z, \tau) H_{r,c_1}^\mu(z, \tau; S, J),$$

with

$$B_{r,k}(z, \tau) = \frac{1}{\eta(\tau)^r} \sum_{\substack{\sum_{i=1}^r a_i = 0 \\ a_i \in \mathbb{Z} + \frac{k}{r}}} q^{-\sum_{i < j} a_i a_j} w^{\sum_{i < j} a_i - a_j}.$$

The two relevant cases for this article are $r = 2, 3$:

$$(6.1) \quad B_{2,k}(z, \tau) = \frac{\sum_{n \in \mathbb{Z} + k/2} q^{n^2} w^n}{\eta(\tau)^2}, \quad B_{3,k}(z, \tau) = \frac{\sum_{m,n \in \mathbb{Z} + k/3} q^{m^2 + n^2 + mn} w^{4m + 2n}}{\eta(\tau)^3}.$$

Note that $B_{r,k}(z, \tau)$ does not depend on S or J .

The computation of $h_{r,c_1}(z, \tau; \mathbb{P}^2)$ from $h_{r,\phi^*c_1-kC}(z, \tau; \Sigma_1)$ in general involves the following three steps:

- (1) Compute $h_{r,\phi^*c_1-kC}^\mu(z, \tau; J_{1,0})$ by adding to $h_{r,\phi^*c_1-kC}(z, \tau; J_{1,\varepsilon})$ terms due to sheaves on Σ_1 which are not Gieseker stable for $J_{1,\varepsilon}$, but μ -semistable for $\phi^*H = J_{1,0}$, and consequently compute $H_{r,\phi^*c_1-kC}^\mu(z, \tau; J_{1,0})$ by adding the terms prescribed by Eq. (3.5). The generating functions and the factorial factors in Eq. (3.5) combine these two steps very naturally into one.
- (2) Divide by $B_{r,k}(z, \tau)$ to obtain $H_{r,c_1}^\mu(z, \tau; \mathbb{P}^2)$.
- (3) Determine $h_{r,c_1}(z, \tau; \mathbb{P}^2)$ from $H_{r,c_1}^\mu(z, \tau; \mathbb{P}^2)$ by reversing step (1).

For $c_1 = \beta C + f$, $\beta = 0$ or 1 , and $J = J_{1,0}$, μ -stability is equivalent to Gieseker stability, and therefore steps 1) and 3) become trivial. For example, one can compute $h_{3,H}(z, \tau; \mathbb{P}^2)$ either starting from $h_{3,f}(z, \tau; J_{1,0})$ (which requires Conjecture 4.1 and Eq. (5.8)) or $h_{3,C+f}(z, \tau; J_{1,0})$. One can verify that the first terms of both q -expansions of $h_{3,H}(z, \tau; \mathbb{P}^2)$ are equal, which is in agreement with Proposition 6.1. A proof of the equality of these expressions for $h_{3,H}(z, \tau; \mathbb{P}^2)$ would imply a proof of Conjecture 4.1 for $(r, c_1) = (3, f)$ since $h_{3,f}(z, \tau; J_{\varepsilon,1})$ is related to $h_{3,C+f}(z, \tau; J_{1,\varepsilon})$ by the blow-up formula and wall-crossing.

When μ - and Gieseker stability are not equivalent, steps 1) and 3) are not trivial. We will first explain them for $r = 2$ following [29]. One obtains:

Proposition 6.2.

$$h_{2,0}(z, \tau; \mathbb{P}^2) = \frac{1}{B_{2,1}(z, \tau)} \left[h_{2,C}(z, \tau; J_{1,\varepsilon}) + \sum_{\substack{b < 0 \\ b \equiv -1 \pmod{2}}} w^b q^{\frac{1}{4}b^2} h_{1,0}(z, \tau)^2 \right] - \frac{1}{2} h_{1,0}(z, \tau; \mathbb{P}^2)^2.$$

Proof. The only extended HN-filtrations which are Gieseker unstable for $J = J_{1,\varepsilon}$ and μ -semi-stable for $J = J_{1,0}$ have $\ell = \ell_\mu = 2$. For the parametrization $c_1(E_2) = bC - af$, the set of sheaves which is unstable for $J_{1,\varepsilon}$ but μ -semistable for $J_{1,0}$ corresponds to $b < 0$ and $a = 0$. This gives the second term inside the brackets. Consequently, step (2) divides by $B_{2,1}(z, \tau)$, and step (3) subtracts the μ -semi-stable sheaves which are not Gieseker semi-stable with $\ell = 2$ and $\ell_\mu = 1$. \square

Alternatively, one can compute $h_{2,0}(z, \tau; \mathbb{P}^2)$ starting from $h_{2,0}(z, \tau; J_{1,\varepsilon})$. In that case the term due to step (1) in the brackets is $\left(\sum_{\substack{b < 0 \\ b \equiv 0 \pmod{2}}} w^b q^{\frac{1}{4}b^2} + \frac{1}{2} \right) h_{1,0}(z, \tau)^2$, and one divides by $B_{2,0}(z, \tau)$. Addition of $\frac{1}{2} h_{1,0}(z, \tau; \mathbb{P}^2)$ provides the expected integer invariants, in agreement with [29]. Accidentally, the terms due to step (1) and step (3) can simply be incorporated by replacing $J_{1,\varepsilon}$ by $J_{1,0}$ in $h_{2,\beta C}(z, \tau; J_{1,\varepsilon})$, and can be written in terms of the Lerch sum [2].

The remainder of this section discusses with $r = 3$. In terms of $h_{3,C}(z, \tau; J_{1,\varepsilon})$, $h_{3,0}(z, \tau; \mathbb{P}^2)$ is given by:

Proposition 6.3.

$$\begin{aligned} h_{3,0}(z, \tau; \mathbb{P}^2) &= \frac{1}{B_{3,1}(z, \tau)} \left[h_{3,C}(z, \tau; J_{1,\varepsilon}) + \left(\sum_{\substack{b < 0 \\ b \equiv -2, -4 \pmod{6}}} w^b q^{\frac{1}{12}b^2} \right) h_{1,0}(z, \tau) h_{2,0}(z, \tau; J_{1,\varepsilon}) \right. \\ (6.2) \quad &+ \left(\sum_{\substack{b < 0 \\ b \equiv -1, -5 \pmod{6}}} w^b q^{\frac{1}{12}b^2} \right) h_{1,0}(z, \tau) h_{2,C}(z, \tau; J_{1,\varepsilon}) \\ &+ \left(\sum_{\substack{k_1, k_2 < 0 \\ k_2 \equiv k_1 + 1 \pmod{3}}} w^{2(k_1 + k_2)} q^{\frac{1}{3}(k_1^2 + k_2^2 + k_1 k_2)} + \frac{1}{2} \sum_{\substack{k < 0, \\ k \equiv -1, -2 \pmod{3}}} w^{2k} q^{\frac{1}{3}k^2} \right) h_{1,0}(z, \tau)^3 \Big] \\ &- \frac{1}{6} h_{1,0}(z, \tau; \mathbb{P}^2)^3 - \frac{2}{2} h_{1,0}(z, \tau; \mathbb{P}^2) h_{2,0}(z, \tau; \mathbb{P}^2). \end{aligned}$$

The desired integer invariants are obtained from $h_{3,0}(z, \tau; \mathbb{P}^2)$ after subtraction of $\frac{1}{3}h_{1,0}(3z, 3\tau; \mathbb{P}^2) = \frac{1}{3} \frac{i}{\theta(6z, 3\tau)}$. The first non-vanishing coefficients are presented in Table 1.² They are in agreement with the expected dimension of $\mathcal{M}(\Gamma)$ (2.3).

c_2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}	b_{14}	b_{16}	b_{18}	b_{20}	b_{22}	b_{24}	b_{26}	b_{28}	χ
3	1	1	2	2	2	2										18
4	1	2	5	9	15	19	22	23	24							216
5	1	2	6	12	25	43	70	98	125	142	154	156				1512
6	1	2	6	13	28	53	99	165	264	383	515	631	723	774	795	8109

TABLE 1. The Betti numbers b_n (with $n \leq \dim_{\mathbb{C}} \mathcal{M}$) and the Euler number χ of the moduli spaces of semi-stable sheaves on \mathbb{P}^2 with $r = 3$, $c_1 = 0$, and $3 \leq c_2 \leq 6$.

Proof. The terms added to $h_{3,C}(z, \tau; J_{1,\varepsilon})$ in the brackets are due to step (1). The last term on the first line and the term on the second line are due to filtrations with $\ell = \ell_{\mu} = 2$. If one chooses $c_1(E_2) = bC - af$ as for $r = 2$, the set of sheaves which are unstable for $J_{1,\varepsilon}$ but μ -semistable for $J_{1,0}$ corresponds to $b < 0$ and $a = 0$. Similarly, the first term in parentheses on the third line is due to $\ell = \ell_{\mu} = 3$, and the second term due to $\ell = 3$ and $\ell_{\mu} = 2$. The sum of the terms in the bracket is $H_{3,C}^{\mu}(z, \tau; J_{1,\varepsilon})$, and is divided by $B_{3,1}(z, \tau)$ following step (2). Finally, step (3) corresponds to the last line. \square

In a similar manner, $h_{3,0}(z, \tau; \mathbb{P}^2)$ can be computed from $h_{3,0}(z, \tau; J_{1,\varepsilon})$. Then the terms due to step (1) are for $\ell = 2$:

$$(6.3) \quad \left(\frac{2}{2} + 2 \sum_{\substack{b < 0 \\ b \equiv 0 \pmod{6}}} w^b q^{\frac{1}{12}b^2} \right) h_{1,0}(z, \tau) h_{2,0}(z, \tau; J_{1,\varepsilon}) \\ + \left(2 \sum_{\substack{b < 0 \\ b \equiv -3 \pmod{6}}} w^b q^{\frac{1}{12}b^2} \right) h_{1,0}(z, \tau) h_{2,C}(z, \tau; J_{1,\varepsilon}),$$

and for $\ell = 3$:

$$(6.4) \quad \left(\sum_{\substack{k_1, k_2 < 0 \\ k_1 = k_2 \pmod{3}}} w^{2(k_1+k_2)} q^{\frac{1}{3}(k_1^2+k_2^2+k_1k_2)} + \frac{2}{2} \sum_{\substack{k < 0 \\ k \equiv 0 \pmod{3}}} w^{2k} q^{\frac{1}{3}k^2} + \frac{1}{6} \right) h_{1,0}(z, \tau)^3.$$

²Note that these numbers are different from Refs. [17, 27], where Euler numbers of moduli spaces of sheaves and vector bundles are computed using localisation methods. This difference might be due to different choices of stability conditions.

REFERENCES

- [1] M. F. Atiyah, R. Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. Roy. Soc. Lond. **A308** (1982) 523-615.
- [2] K. Bringmann and J. Manschot, *From sheaves on \mathbb{P}^2 to a generalization of the Rademacher expansion*, Am. J. of Math., arXiv:1006.0915 [math.NT].
- [3] U. V. Desale and S. Ramanan, *Poincaré polynomials of the variety of stable bundles*, Math. Ann. **216** (1975) 233-244.
- [4] E. Diaconescu, G. W. Moore, *Crossing the wall: Branes versus bundles*, [arXiv:0706.3193 [hep-th]].
- [5] R. Friedman, “Algebraic Surfaces and Holomorphic Vector Bundles,” Springer-Verlag (1998).
- [6] D. Gaiotto, G. W. Moore and A. Neitzke, *Framed BPS States*, arXiv:1006.0146 [hep-th].
- [7] L. Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. **286** (1990) 193.
- [8] L. Göttsche, D. Zagier, *Jacobi forms and the structure of Donaldson invariants for 4-manifolds with $b_+ = 1$* , Selecta Math., New Ser. **4** (1998) 69. [arXiv:alg-geom/9612020].
- [9] L. Göttsche, *Theta functions and Hodge numbers of moduli spaces of sheaves on rational surfaces*, Comm. Math. Physics **206** (1999) 105 [arXiv:math.AG/9808007].
- [10] G. Harder, M. S. Narasimhan, *On the cohomology groups of moduli spaces of vector bundles on curves*, Math. Ann. **212** (1975) 215-248.
- [11] D. Huybrechts and M. Lehn, “The geometry of moduli spaces of sheaves,” Cambridge University Press (1996).
- [12] D. Joyce, *Configurations in Abelian categories. IV. Invariants and changing stability conditions*, [arXiv:math.AG/0410268].
- [13] D. Joyce and Y. Song, *A theory of generalized Donaldson-Thomas invariants*, arXiv:0810.5645 [math.AG].
- [14] H. Kim, J. Park, Z. Wang, P. Yi, *Ab Initio Wall-Crossing*, [arXiv:1107.0723 [hep-th]].
- [15] F. Kirwan, *On the homology of compactifications of moduli spaces of vector bundles over Riemann surfaces*, Proc. London Math. Soc. **53** (1986) 237-266; *Corrigendum*, Proc. London Math. Soc. **65** (1992) 474.
- [16] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, [arXiv:0811.2435 [math.AG]].
- [17] M. Kool, *Euler characteristics of moduli spaces of torsion free sheaves on toric surfaces*, arXiv:0906.3393 [math.AG].
- [18] W.-P. Li and Z. Qin, *On blowup formulae for the S-duality conjecture of Vafa and Witten*, Invent. Math. **136** (1999) 451-482 [arXiv:math.AG/9808007].
- [19] J. Manschot, *Stability and duality in $\mathcal{N} = 2$ supergravity*, Commun. Math. Phys. **299** (2010) 651-676, arXiv:0906.1767 [hep-th].
- [20] J. Manschot, *Wall-crossing of D4-branes using flow trees*, Adv. Theor. Math. Phys. **15** (2011) 1-42, arXiv:1003.1570 [hep-th].
- [21] J. Manschot, *The Betti numbers of the moduli space of stable sheaves of rank 3 on \mathbb{P}^2* , Lett. in Math. Phys. **98** (2011) 65-78, [arXiv:1009.1775 [math-ph]].
- [22] J. Manschot, B. Pioline, A. Sen, *Wall Crossing from Boltzmann Black Hole Halos*, JHEP **1107** (2011) 059. [arXiv:1011.1258 [hep-th]].
- [23] J. Manschot, *BPS invariants of $\mathcal{N} = 4$ gauge theory on a surface*, [arXiv:1103.0012 [math-ph]].
- [24] H. Nakajima and K. Yoshioka, *Instanton counting and Donaldson invariants*, Sugaku Expositions **23** (2010) 2.
- [25] M. Reineke, *The Harder-Narasimhan system in quantum groups and cohomology of quiver moduli*, Invent. Math. **152** (2003) 349.
- [26] C. Vafa and E. Witten, *A strong coupling test of S duality*, Nucl. Phys. B **431** (1994) 3 [arXiv:hep-th/9408074].

- [27] T. Weist, *Torus fixed points of moduli spaces of stable bundles of rank three*, arXiv:0903.0732 [math. AG].
- [28] K. Yoshioka, *The Betti numbers of the moduli space of stable sheaves of rank 2 on \mathbb{P}^2* , J. reine. angew. Math. **453** (1994) 193–220.
- [29] K. Yoshioka, *The Betti numbers of the moduli space of stable sheaves of rank 2 on a ruled surface*, Math. Ann. **302** (1995) 519–540.
- [30] K. Yoshioka, *The chamber structure of polarizations and the moduli of stable sheaves on a ruled surface*, Int. J. of Math. **7** (1996) 411–431 [arXiv:alg-geom/9409008].
- [31] K. Yoshioka, *Euler characteristics of $SU(2)$ instanton moduli spaces on rational elliptic surfaces*, Commun. Math. Phys. **205** (1999) 501 [arXiv:math/9805003].
- [32] D. Zagier, *Elementary aspects of the Verlinde formula and of the Harder-Narasimhan-Atiyah-Bott formula*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), 445–462, Israel Math. Conf. Proc., 9, Bar-Ilan Univ., Ramat Gan, 1996.

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